

The role of retardation in 3-D relativistic equations

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Abstract

Equal-time Green's function is used to derive a three-dimensional integral equation from the Bethe-Salpeter equation. The resultant equation, in the absence of anti-particles, is identical to the use of time-ordered diagrams, and has been used within the framework of $\phi^2\sigma$ coupling to study the role of energy dependence and non-locality when the two-body potential is the sum of σ -exchange and crossed σ exchange. The results show that non-locality and energy dependence make a substantial contribution to both the on-shell and off-shell amplitudes.

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I. INTRODUCTION

With the advent of Quantum Chromodynamics (QCD) as the fundamental underlying theory of strong interaction, it would be nice to relate the Lagrangian used in the nucleon-nucleon interactions with a meson-baryon effective Lagrangian extracted from QCD. Although it is not possible at present to write such an effective Lagrangian with all the coupling of the meson to the baryon predetermined by QCD, we can start with a Lagrangian that preserves the symmetries of QCD, *e.g.* chiral symmetry. At this stage the coupling constants will have physical significance in that they can be related to QCD parameters. On the other hand, a determination of the coupling constants in an effective chiral Lagrangian [1] from the experimental data, *e.g.* the nucleon-nucleon phase shifts, could be used to test models of QCD.

Modern nucleon-nucleon interactions [2,3] based on meson exchange have achieved the remarkable success of fitting the ‘experimental’ phase shifts with a χ^2 per data of approximately one [3]. These interactions invariably start with a Lagrangian that includes the coupling of the nucleon to a set of mesons, with the coupling constants of the mesons to the nucleon as parameters to be adjusted to fit the experimental data. To determine the phase shifts for a given Lagrangian, we need to: (i) Define the two-body equation to be solved for the scattering amplitude. (ii) Define the kernel or potential for this two-body equation. In an ideal world, to maintain covariance, we need to solve the Bethe-Salpeter (BS) equation [4] with the kernel being the sum of all two-particle irreducible Feynman diagrams for nucleon-nucleon scattering. In practice this is not simple, and the standard procedure has been: (i) To replace the four-dimensional Bethe-Salpeter equation by a three-dimensional (3-D) equation that reduces to the Lippmann-Schwinger equation [5] in the non-relativistic limit. (ii) Approximate the potential for this 3-D equation by the sum of all single meson exchanges or one- plus two-meson exchanges. (iii) To use the resultant potential in three or more nucleon systems (*i.e.* the Schrödinger equation), reduce the momentum and energy dependence of the potential in order to generate a local coordinate space potential.

The aims of the present investigation are to test, in a scalar theory, the effect on the coupling constant of: (i) Replacing the Bethe-Salpeter equation by a corresponding 3-D equation. (ii) The subtraction of the anti-particles’ contributions from the kernel. (ii) The removal of the non-locality and energy dependence in the potential to allow for a coordinate space local potential. This will give us a qualitative measure of the error in the coupling constant resulting from the standard approximation used in generating local nucleon-nucleon potentials.

In principle there are an infinity of relativistic 3-D equations that satisfy the same unitarity conditions as the Bethe-Salpeter equations [6]. However, in practice there are four equations most commonly used in nucleon-nucleon scattering. Three result as a direct reduction of the Bethe-Salpeter equation. These are: (i) The Blankenbecler-Sugar [7] equation in which the relative energy is set to zero. (ii) The Gross [8] equation in which one of the particles is on-mass-shell. (iii) The Klein [9] equation in which the relative energy is integrated out. This latter equation has been used in recent years in conjunction with time ordered perturbation theory to determine the two-meson exchange nucleon-nucleon potential. The difference between these three equations is the treatment of the off-mass-shell degree of freedom present in the BS equation. The fourth equation is based on Hamiltonian dynamics

[10] with the connection to the field theoretic Lagrangian made via the Okubo projection method [11] as implemented by Fuda [12]. In this equation the nucleons are on-mass-shell.

In Sec. II we will derive the Klein equation from the equal-time Green's function [13], and in this way establish a systematic procedure for improvement on the standard potential in the Klein equation as suggested by Phillips and Wallace [14]. Although we have reduced the dimensionality of the equation from 4-D to 3-D, the kernel still includes the contribution from anti-particles. In Sec. III we examine the kernel of the Klein equation. Here we find that the contribution of anti-particles, at the level of one meson exchange, can be significant for sufficiently large coupling constants. At a coupling constant that gives a binding energy comparable to that of the deuteron, the contribution from anti-particles can be neglected. The neglect of the anti-particle contribution reduces the kernel of the Klein equation to that resulting from time-ordered perturbation theory as has been implemented for the nucleon-nucleon system in both the Bonn and Nijmegen potentials. We then proceed to define the potential arising from the exchange of two mesons in the no-anti-particle (NAP) approximation. In the Klein approach, with NAP, the one- plus two-meson exchange potential is both non-local and energy dependent, and to derive the corresponding local potential we need to remove these dependencies. In Sec. IV we compare the results of the BS, Klein-NAP and the energy independent potentials with different levels of non-locality. We find that although the BS and Klein-NAP are in reasonable agreement, the removal of non-locality and energy dependence could be too severe. The effect of these approximations and the introduction of a form factor on the value of the coupling constant and the off-shell behavior of the potential is examined. Finally, in Sec. V we discuss some of the possible implications of our results for the nucleon-nucleon interaction.

II. THEORY

To establish the relation between the BS and Klein equations, we derive the Klein equation [9] from the equal-time Green's function [13]. In this way, we establish the relation between the four dimensional Bethe-Salpeter equation [4] and the three dimensional Klein equation in the form recently proposed by Phillips and Wallace [14] which allows for a systematic improvement in the kernel of the Klein equation to reproduce the result of the BS equation.

The two-body Green's function or four-point function is given in terms of the vacuum expectation value of the time ordered product of fields as [15]:

$$G(x, y; x', y') \equiv \langle 0 | T(\psi(x) \psi(y) \bar{\psi}(x'), \bar{\psi}(y') | 0 \rangle \quad (1)$$

The equal-time Green's function is then defined as [13]:

$$G(x_0 \mathbf{x}, x_0 \mathbf{y}; x'_0 \mathbf{x}', x'_0 \mathbf{y}') \equiv \int dy_0 dy'_0 \delta(x_0 - y_0) \delta(x'_0 - y'_0) G(x, y; x', y') \quad (2)$$

where we have written the four-vectors using the notation $x = (x_0, \mathbf{x})$. To simplify matters, we need to write the coordinate space Green's function in terms of the momentum space Green's function. This can be achieved by first introducing the relative and center of mass momenta as:

$$k = \frac{1}{2}(p_1 - p_2) \quad \text{and} \quad P = p_1 + p_2 \quad (3)$$

then $d^4 p_1 d^4 p_2 = d^4 P d^4 k$. The equal-time Green's function can now be written as

$$G(x_0 \mathbf{x}, x_0 \mathbf{y}; x'_0 \mathbf{x}', x'_0 \mathbf{y}') = \int d^4 P d^3 k d^3 k' e^{iP_0(x_0 - x'_0)} e^{-i\mathbf{P} \cdot [(\mathbf{x} + \mathbf{y})/2 - (\mathbf{x}' + \mathbf{y}')/2]} \\ \times e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} \tilde{G}(\mathbf{k}, \mathbf{k}'; P) e^{i\mathbf{k}' \cdot (\mathbf{x}' - \mathbf{y}')} \quad (4)$$

where $\tilde{G}(\mathbf{k}, \mathbf{k}'; P)$, the ‘equal-time’ Green's function in momentum space, is given by

$$\tilde{G}(\mathbf{k}, \mathbf{k}'; P) = \int_{-\infty}^{+\infty} dk_0 dk'_0 G(k_0 \mathbf{k}, k'_0 \mathbf{k}'; P) \equiv \langle G \rangle . \quad (5)$$

We note that this ‘equal-time’ Green's function is *not* a function of the ‘relative energy’ k_0 , and has the potential of setting the framework for a three-dimensional integral equation for both the Green's function and the scattering amplitude or T -matrix.

The two-body Green's function defined in Eq. (1) satisfies the BS equation

$$G = G_0 + G_0 K G \\ = G_0 + G_0 T G_0 \quad (6)$$

where G_0 is the free two-particle Green's function, the product of the Feynman propagators for the two particles, and T is the T -matrix for two-particle scattering. In Eq. (6), K is the potential in the BS equation and consists of the sum of all two-particle irreducible Feynman diagrams that contribute to the Green's function. Using the second line in Eq. (6) to iterate the first line, we can write the BS equation for the T -matrix as

$$T = K + K G_0 T . \quad (7)$$

The aim of this section is to write an approximation to Eq. (7) that is an integral equation in three-dimensions. There are an infinity of such equations [6], and to find the optimum one is not the aim of the present study.

The unique feature of the Klein equation is that the potential is assumed to be independent on the ‘relative’ energy k_0 , and as a result the ‘relative’ energy integration, in the integral equation, can be carried through resulting in a 3-D equation. This suggests that one may develop an approximation scheme based on the idea that the potential K can be divided in two parts [14]. The first part K_1 is independent of the ‘relative’ energy, i.e.,

$$K_1(k, k'; P) = K_1(\mathbf{k}, \mathbf{k}'; P) , \quad (8)$$

and the rest, $K_2 = K - K_1$, has all the ‘relative’ energy dependence. The problem now reduces to finding an optimum K_1 such that the solution to the 3-D integral equation is a good approximation to the solution of the BS equation, and to define a systematic procedure for improving the results of the Klein equation.

Let us first assume that $K_2 = 0$, then the ‘equal-time’ Green's function can be written as

$$\begin{aligned}
\langle G \rangle &= \langle G_0 \rangle + \langle G_0 T G_0 \rangle \\
&= \langle G_0 \rangle + \langle G_0 K_1 G \rangle ,
\end{aligned} \tag{9}$$

where

$$\langle A \rangle \equiv \tilde{A}(\mathbf{k}, \mathbf{k}'; P) \equiv \int_{-\infty}^{+\infty} dk_0 dk'_0 A(k, k'; P) . \tag{10}$$

However, since K_1 does not depend on the ‘relative’ energy, it is simple to show that $\langle G_0 K_1 G \rangle = \langle G_0 \rangle K_1 \langle G \rangle$ and as a result we can write

$$\begin{aligned}
\langle G_0 T G_0 \rangle &= \langle G_0 \rangle K_1 \langle G \rangle \\
&= \langle G_0 \rangle K_1 \{ \langle G_0 \rangle + \langle G_0 T G_0 \rangle \} .
\end{aligned} \tag{11}$$

This allows us to write a 3-D integral equation, the Klein equation, in which K_1 is the potential, as

$$T_1 = K_1 + K_1 \langle G_0 \rangle T_1 , \tag{12}$$

where the amplitude T_1 is the ‘equal-time’ T -matrix, and is defined in terms of the BS T -matrix as

$$T_1 \equiv \langle G_0 \rangle^{-1} \langle G_0 T G_0 \rangle \langle G_0 \rangle^{-1} . \tag{13}$$

At this stage all we have established is that if K_1 is independent of the relative energy, then we have the 3-D equation first proposed by Klein [9].

The potential in the BS equation is the sum of all two-particle irreducible diagrams that contribute to the amplitude, and in general this potential depends on the ‘relative’ energy. In this case K_2 is not zero, and we need to define an optimum K_1 . The Green’s function for the potential K_2 is given by

$$G_{K_2} = G_0 + G_0 K_2 G_{K_2} = G_0 + G_0 (K - K_1) G_{K_2} , \tag{14}$$

and the corresponding ‘equal-time’ Green’s function is given by

$$\langle G_{K_2} \rangle = \langle G_0 \rangle + \langle G_0 K G_{K_2} \rangle - \langle G_0 \rangle K_1 \langle G_{K_2} \rangle . \tag{15}$$

If we now define K_1 , the potential that is independent of the ‘relative energy’, in terms of the kernel of the BS equation as

$$K_1 \equiv \langle G_0 \rangle^{-1} \langle G_0 K G_{K_2} \rangle \langle G_0 \rangle^{-1} , \tag{16}$$

then Eq. (15) can be solved for the ‘equal-time’ Green’s function for the potential K_2 with the simple solution that $\langle G_{K_2} \rangle = \langle G_0 \rangle$ i.e. the ‘equal-time’ Green’s function for the potential K_2 is identical to the ‘equal-time’ free Green’s function, provided K_1 is defined as in Eq. (16).

The Green’s function for the full BS kernel K can now be written in terms of K_2 as

$$\begin{aligned}
G &= G_0 + G_0 T G_0 \\
&= G_{K_2} + G_{K_2} K_1 G ,
\end{aligned} \tag{17}$$

with the corresponding ‘equal-time’ Green’s function given by

$$\begin{aligned}\langle G \rangle &= \langle G_0 \rangle + \langle G_0 T G_0 \rangle \\ &= \langle G_{K_2} \rangle + \langle G_{K_2} K_1 G \rangle .\end{aligned}\tag{18}$$

We are now in a position to write the ‘equal-time’ amplitude as

$$\begin{aligned}\langle G_0 T G_0 \rangle &= \langle G_{K_2} K_1 G \rangle = \langle G_{K_2} \rangle K_1 \langle G \rangle \\ &= \langle G_0 \rangle K_1 \{ \langle G_0 \rangle + \langle G_0 T G_0 \rangle \} .\end{aligned}\tag{19}$$

This result again gives us the Klein equation as defined in Eq. (12) with the potential in this case given by Eq. (16). Clearly, we cannot determine this potential for the full BS kernel with G_{K_2} a solution of Eq. (14). This would be equivalent to solving the full BS equation with no approximations to the kernel. As a first approximation, we could expand the kernel of the BS equation in powers of the coupling constant, and at the same time iterate Eq. (14) to keep all terms of the same order as in the BS kernel. This will effectively give us an expansion of the potential in the Klein equation as a power series in the coupling constant. We will see in the next section, for Yukawa type coupling, that to order g^2 this potential corresponds to single meson exchange, while to order g^4 , K_1 will include one- and two-meson exchanges. The question then is: how do the results of this three-dimensional equation (the Klein equation) compare with the results from the BS equation when the potential is calculated to the same order?

The definition of the ‘equal-time’ potential in Eq. (16) allows for a systematic way of calculating the amplitude within the framework of a three-dimensional equation. In addition, this equation includes both the positive and negative energy component of the Feynman propagators. This result is identical to that used by Phillips and Wallace for the two-body bound state problem [14]. One can show that in the absence of negative energy states, this expression for the potential is equivalent to the result of time-ordered perturbation theory [18,19]. Since the S -matrix resulting Eq. (12) with K_1 defined in Eq. (16) is identical to the S -matrix from the original BS equation, we have maintained covariance, but it is not manifest covariance. Furthermore, since truncation in both K and G_{K_2} are carried through at the field theory level, we expect covariance to be maintained.

Finally, to fourth order in the coupling constant, and in the absence of anti-particles, this potential is identical to that used to include two pion exchange in the nucleon-nucleon interaction before any approximation is required to transform the potential to coordinate space [16,17]. This therefore can form the basis of estimating the approximate magnitude of the errors made in going from the BS equation to a coordinate space local potential used in the Schrödinger equation for a given Lagrangian.

III. THE TWO-BODY POTENTIAL

To examine the approximations needed to reduce the kernel of the BS equation to a local potential in coordinate space, we need to define a Lagrangian. To avoid the problems with spin and isospin, especially at the BS level with crossed meson exchange, we have considered a $\phi^2\sigma$ theory, *i.e.* a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} \left[\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 \right] + \frac{1}{2} \left[\partial^\mu \sigma \partial_\mu \sigma - \mu^2 \sigma^2 \right] - g \phi^2 \sigma . \quad (20)$$

In the nucleon-nucleon interaction the kernel of the BS equation is truncated to include the one pion exchange and the crossed two pion exchange. To compensate for this truncation one includes the exchange of all mesons with a mass less than 1 GeV. The motivation is that the heavy meson exchange will model all higher order diagrams that have been excluded [20]. With this approximation in mind, we make use of the above Lagrangian to define the kernel of the BS equation in terms of σ -exchange and the crossed σ -exchange. In addition, we will define the potential in the Klein equation to be an approximation to K_1 , and the order of this approximation to be determined by our approximation to the kernel of the BS equation, *i.e.* the potential in both equations is taken to the same order in the coupling constant. In this way we expect the mechanism included in both the BS and Klein equations to be the same, and the difference is the result of using a 3-D equation in place of the 4-D equation.

With the above scheme in mind, we first consider the potential in the Klein equation to order g^2 . In this case we include the kernel of BS equation to order g^2 and take $G_{K_2} \approx G_0$. This gives the single σ -exchange potential as:

$$K_1^{(2)} = \langle G_0 \rangle^{-1} \langle G_0 I^{(2)} G_0 \rangle \langle G_0 \rangle^{-1} , \quad (21)$$

where $I^{(2)}$, the σ -exchange amplitude as employed in the BS equation, is given by

$$I^{(2)}(k, k') = \frac{g^2}{(k - k')^2 - \mu^2 + i\epsilon} , \quad (22)$$

with the relative four-momentum $k = (k_0, \mathbf{k})$. To evaluate the ‘equal-time’ matrix elements $\langle G_0 \rangle$ and $\langle G_0 I^{(2)} G_0 \rangle$, we need to perform the relative energy integration in both the initial and final states (see Eq. (10)). For this we decompose the product of the Feynman propagators for the two ϕ fields, in the center of mass, in terms of their positive and negative energy components, *i.e.*

$$\begin{aligned} G_0(k, P) &= \frac{i}{(2\pi)^4} \frac{1}{(\frac{\sqrt{s}}{2} + k_0)^2 - E_k^2 + i\epsilon} \frac{1}{(\frac{\sqrt{s}}{2} - k_0)^2 - E_k^2 + i\epsilon} , \\ &= G_0^{++} + G_0^{+-} + G_0^{-+} + G_0^{--} . \end{aligned} \quad (23)$$

By performing the relative energy integration, we reduce the ‘equal-time’ free Green’s function to:

$$\begin{aligned} \langle G_0 \rangle &= \frac{1}{(2\pi)^3} \frac{1}{E_k} \frac{1}{s - (2E_k)^2 + i\epsilon} \\ &= \langle G_0^{++} \rangle + \langle G_0^{--} \rangle , \end{aligned} \quad (24)$$

where

$$\begin{aligned} \langle G_0^{++} \rangle &= \frac{1}{(2\pi)^3} \frac{1}{(2E_k)^2} \frac{1}{\sqrt{s} - 2E_k + i\epsilon} \\ \langle G_0^{--} \rangle &= -\frac{1}{(2\pi)^3} \frac{1}{(2E_k)^2} \frac{1}{\sqrt{s} + 2E_k - i\epsilon} . \end{aligned} \quad (25)$$

We note here that the free ‘equal-time’ Green’s function is a function $s = P^2$ and not \sqrt{s} as in a number of three-dimensional equations. It is only when we divide this Green’s function into its positive and negative energy components, and neglect the negative energy component, do we get the dependence on \sqrt{s} .

To perform the relative energy integration on the Green’s function for σ -exchange, (*i.e.* $G_0 I^{(2)} G_0$) we again take advantage of the decomposition of the Green’s function in terms of a positive and negative energy component to write

$$\begin{aligned} \langle G_0 I^{(2)} G_0 \rangle &= \int_{-\infty}^{+\infty} dk_0 dk'_0 G_0(k, P) I^{(2)}(k, k') G_0(k', P) \\ &= \sum_{\alpha\beta} \sum_{\gamma\delta} I_{\alpha\beta;\gamma\delta}^{(2)}(\mathbf{k}, \mathbf{k}'; P) \end{aligned} \quad (26)$$

where

$$I_{\alpha\beta;\gamma\delta}^{(2)}(\mathbf{k}, \mathbf{k}'; P) = \langle G_0^{\alpha\beta} I^{(2)} G_0^{\gamma\delta} \rangle, \quad (27)$$

with $\alpha, \beta, \gamma, \delta = +, -$. Making use of the symmetry of the integrals $I_{\alpha\beta;\gamma\delta}^{(2)}(\mathbf{k}, \mathbf{k}'; P)$ under the exchange of indices and momenta to reduce the number of integrals, we can write the ‘equal-time’ Green’s function for σ -exchange as:

$$\begin{aligned} \langle G_0 I^{(2)} G_0 \rangle &= \left[\langle G_0^{++} \rangle + \Delta \right] d_+ \left[\langle G_0^{++} \rangle + \Delta \right] \\ &\quad + \left[\langle G_0^{++} \rangle + \Delta \right] d_+ \left[\langle G_0^{++} \rangle + \Delta \right] \\ &\quad + \left[\langle G_0^{--} \rangle + \Delta \right] d_- \left[\langle G_0^{--} \rangle + \Delta \right] \\ &\quad + \left[\langle G_0^{--} \rangle + \Delta \right] d_- \left[\langle G_0^{--} \rangle + \Delta \right] \\ &\quad - 2 \langle G_0^{++} \rangle d_0 \langle G_0^{--} \rangle - 2 \langle G_0^{--} \rangle d_0 \langle G_0^{++} \rangle, \end{aligned} \quad (28)$$

where

$$\begin{aligned} d_+(\mathbf{k}, \mathbf{k}') &= \frac{g^2}{2\omega} \frac{1}{\sqrt{s} - E_k - E_{k'} - \omega + i\epsilon} \\ d_-(\mathbf{k}, \mathbf{k}') &= -\frac{g^2}{2\omega} \frac{1}{\sqrt{s} + E_k + E_{k'} + \omega - i\epsilon} \\ d_0(\mathbf{k}, \mathbf{k}') &= \frac{g^2}{2\omega} \frac{1}{E_k + E_{k'} + \omega}, \end{aligned} \quad (29)$$

and

$$\Delta(\mathbf{k}, \mathbf{k}') = -\frac{1}{(2\pi)^3} \frac{1}{(2E_k)^2} \frac{1}{E_k + E_{k'} + \omega}. \quad (30)$$

This result is identical to that of Phillips and Wallace [14]. We observe here that the elimination of the negative energy components of the ‘equal-time’ free Green’s function, *i.e.* $\langle G_0 \rangle \rightarrow \langle G_0^{++} \rangle$ reduces the above expression for the ‘equal-time’ Green’s function to terms with $\langle G_0^{++} \rangle$ only, *i.e.*

$$\langle G_0 I^{(2)} G_0 \rangle = 2 \langle G_0^{++} \rangle d_+ \langle G_0^{++} \rangle . \quad (31)$$

As a result, the potential in the Klein equation reduces to the simple form used in time ordered perturbation theory:

$$K_1^{(2)}(\mathbf{k}, \mathbf{k}'; s) \approx \frac{g^2}{\omega} \frac{1}{\sqrt{s} - E_k - E_{k'} - \omega + i\epsilon} . \quad (32)$$

We will refer to this approximation as the Klein potential with no anti-particles (Klein-NAP). This major simplification of the potential, in neglecting the negative energy component of the two-particle free Green's function, can have a major reduction in the structure of the crossed two σ exchange contribution.

Before we proceed to the evaluation of potential K_1 to order g^4 , we need to estimate the contribution of the negative energy component of the ϕ - ϕ Green's function. To get our scalar model to be a reasonable approximation to the nucleon-nucleon interaction with pion exchange, we have chosen the mass of the ϕ to be $m = 1.0$ GeV, while the mass of the σ is taken as $\mu = 0.15$ GeV. In this way the range of the interaction is comparable to that generated by one pion exchange. In Fig. 1 we plot the mass of the ϕ - ϕ bound state as a function of the strength of the coupling, *i.e.* $\frac{g^2}{4\pi}$, for the BS, Klein, and Klein-NAP. Here we observe that the negative energy states can make a substantial contribution to the binding energy if the coupling is strong enough. However, if we assume that the binding energy of the ϕ - ϕ system is comparable to that of the deuteron (*i.e.* 2.225 MeV), then the coupling constant is $\frac{g^2}{4\pi} = 1.646$, and the contribution to the binding energy of the negative energy states or anti-particles is negligible. In Fig. 2 we present the phase shifts for the one sigma exchange potential, corresponding to the coupling constant of $\frac{g^2}{4\pi} = 1.646$, for the BS, Klein, and Klein-NAP equations as a function of energy. Here we note that all three results are very close, and that the contribution of the negative energy states is comparable to the difference between the BS and the Klein equations. The fact that the Klein equation gives a good approximation to the BS equation is an indication that one can work within the framework of the three-dimensional Klein equation with no anti-particles for the range of coupling constants that are consistent with the nucleon-nucleon potential.

At this stage we have carried out the determination of the potential for the Klein equation to order g^2 . To include the crossed σ -exchange, we need to go to fourth order in the coupling constant. The starting point is still Eq. (16), but now we have to include all contributions to K to order g^4 as well as the contribution from G_{K_2} . With the help of Eq. (14), we can write K_1 , after one iteration of G_{K_2} , as:

$$\begin{aligned} K_1 &= \langle G_0 \rangle^{-1} \langle G_0 K G_{K_2} \rangle \langle G_0 \rangle^{-1} \\ &\approx \langle G_0 \rangle^{-1} \langle G_0 K [G_0 + G_0 (K - K_1) G_0] \rangle \langle G_0 \rangle^{-1} . \end{aligned} \quad (33)$$

If we now take the kernel of the BS equation to fourth order in the coupling, *i.e.*

$$K \approx I^{(2)} + I^{(4)} , \quad (34)$$

where $I^{(4)}$ is the crossed σ -exchange diagram, and keep terms to fourth order in the coupling, we get

$$\begin{aligned}
K_1 &\approx \langle G_0 \rangle^{-1} \langle G_0 I^{(2)} G_0 \rangle \langle G_0 \rangle^{-1} + \langle G_0 \rangle^{-1} \langle G_0 I_x^{(4)} G_0 \rangle \langle G_0 \rangle^{-1} \\
&\quad + \langle G_0 \rangle^{-1} \left[\langle G_0 I^{(2)} G_0 I^{(2)} G_0 \rangle - \langle G_0 I^{(2)} G_0 K_1^{(2)} G_0 \rangle \right] \langle G_0 \rangle^{-1} \\
&\equiv K_1^{(2)} + K_1^{(4)} .
\end{aligned} \tag{35}$$

The first term is just the ‘equal-time’ single σ -exchange, while the second term is the ‘equal-time’ crossed σ -exchange. In addition to these two contributions, we have a part of the boxed diagram that cannot be represented by the sequential ‘equal-time’ two σ -exchange. This contribution is given in the third and final term, and consists of the ‘equal-time’ boxed diagram minus the once iterated ‘equal-time’ single σ -exchange, since

$$\langle G_0 \rangle^{-1} \langle G_0 I^{(2)} G_0 K_1^{(2)} G_0 \rangle \langle G_0 \rangle^{-1} = K_1^{(2)} \langle G_0 \rangle K_1^{(2)} . \tag{36}$$

In general, the evaluation of this potential, $K_1^{(4)}$, involves the evaluation of the relative energy integration in both initial and final states for both the boxed and crossed σ -exchanges. To facilitate the evaluation of these integrals, we have retained only the positive energy component of the free ϕ - ϕ Green’s function, *i.e.* we have assumed that $G_0 \rightarrow G_0^{++}$ in the evaluation of the potential. This gives us the result of time ordered perturbation theory [16,21], *i.e.* the potential $K_1^{(4)}$ is the sum of the diagrams in Fig. 3. Note that in making use of time ordered perturbation theory we have assumed that the contribution of anti-particles is negligible. That in turn was justified on the basis of our results for the present Lagrangian and at a value of the coupling constant that gives a binding energy for the ϕ - ϕ that is comparable to the deuteron binding energy.

IV. NUMERICAL RESULTS

To examine the approximation required to reduce the BS equation to the non-relativistic Schrödinger equation with a local coordinate space potential, we define the interaction at the level of the BS equation. We will include in the interaction either a single σ -exchange (*i.e.* order g^2) or σ -exchange plus crossed σ -exchange (*i.e.* order g^4). At this stage the only parameter in the potential is the coupling constant g , which we set in the last section to reproduce a $\phi - \phi$ bound state with a binding energy of 2.225 MeV. This fixes the coupling constant at a value of $\frac{g^2}{4\pi} = 1.646$ for single σ -exchange, and $\frac{g^2}{4\pi} = 1.484$ for single σ -exchange and crossed σ exchange. We now carry through a number of approximations to this fully covariant model. These being: (i) At the one σ -exchange level, we compare the results of the BS with the Klein-NAP, and the corresponding potential with no energy dependence. (ii) At the σ -exchange plus crossed σ -exchange, we compare the results of the BS and Klein-NAP with the results of removing first the energy dependence, *i.e.* the retardation, and second the momentum dependence, *i.e.* the non-locality of the potential in the Klein equation.

Since the Klein-NAP approximation is identical to time ordered perturbation theory, the potential has energy dependence of the form exhibited in Eq. (32). This energy dependence corresponds to a retardation in the potential which may cause problems if implemented in many-body calculations. To remove the energy dependence in a typical denominator *e.g.*

$$D = \frac{1}{\sqrt{s} - E_k - E_{k'} - \omega} , \tag{37}$$

we first expand the energy of the ϕ and the total energy of the system in a power series of the inverse of the ϕ mass on the grounds that the typical momenta are small when compared with the ϕ mass, *i.e.*

$$\begin{aligned}\sqrt{s} &= 2\sqrt{k_0^2 + m^2} = 2m + \frac{k_0^2}{m} + \dots \\ E_k &= \sqrt{k^2 + m^2} = m + \frac{k^2}{2m} + \dots,\end{aligned}\tag{38}$$

where k_0 is the on-shell momentum. We then follow Sugawara and Okubo [22] and Rijken [16] and expand D in powers of m^{-1} , with the result that

$$D = -\frac{1}{\omega} \left[1 + \frac{2k_0^2 - k^2 - k'^2}{2m\omega} + \left(\frac{2k_0 - k^2 - k'^2}{2m\omega} \right)^2 + \dots \right].\tag{39}$$

The first term in this expansion, *i.e.* $O(1)$, is referred to as the “adiabatic” approximation, and if we keep terms of $O(m^{-1})$ we have the “non-adiabatic” approximation [16]. Although this approximation might be valid for the \sqrt{s} below the threshold for σ production, the corresponding expansion for $E_{k'}$ can be questioned since k' is an integration variable that takes on all values from zero to infinity in solving the integral equation [23]. The “adiabatic” approximation at the one σ -exchange level is basically the static approximation in which $E_k + E_{k'} = \sqrt{s}$, and in this limit the potential has no energy dependence, and reduces to a local Yukawa potential.

For the two σ -exchange contribution, *i.e.* order g^4 , we get factors of the form given in Eq. (37) for intermediate states. In this case to remove the energy dependence one needs to carry through an “on-energy-shell” approximation which requires that:

$$k^2 - k_0^2 \rightarrow 0 \quad \text{and} \quad k'^2 - k_0^2 \rightarrow 0,\tag{40}$$

where k and k' are the initial and final state momenta respectively. The final form of the resultant potential [16,21] to order g^4 in both the adiabatic and non-adiabatic approximations, has no energy dependence. The momentum dependence then is determined by the expansion in powers of m^{-1} of all energy denominators, and the number of terms kept in such an expansion.

In Fig. 4 we present the s -wave phase shifts for the case when the potential is defined to be due to a single σ -exchange. Here we observe that although the results of the BS and Klein-NAP equations are in reasonably good agreement, the removal of the energy dependence has a substantial effect on the phase shifts. Although this difference could be compensated for by the adjustment of the coupling constant g , it is not clear that the off-energy-shell difference will not persist. To illustrate this, we present in Fig. 5 the Kowalski-Noyes [24,25] half-off-shell function for the Klein-NAP potential (solid line) and the energy independent potential I (dashed line). Also included, is the result for the half-off-shell function for an energy independent potential II (dotted line) that gives almost the same phase shifts as the Klein-NAP potential. Clearly the half-off-shell function for the energy independent and energy dependent potentials are quite different. This is true even when the two potentials have approximately the same phase shifts. Such a large variation in the off-shell behavior of

the amplitude, even when the potentials are identical on-shell, can have significant effects on three- and many-body results [26].

To get a measure of the uncertainty in the coupling constant g as a result of removing the energy dependence of the potential while keeping the physical observables the same, we have adjusted the coupling constant g such that the binding energy of the ϕ - ϕ system is 2.225 MeV in the BS, Klein-NAP and the energy independent potential. The resultant coupling constants are presented in Table I. Here we observe a change of 6% in going from the BS to the Klein-NAP equation as compared with a 13% change in going from the BS to energy independent potential. This change in the coupling constant is significant if the coupling constant is to be a meaningful quantity that might be derived from some underlying quantum field theory, such as QCD for the nucleon-nucleon interaction.

The contribution to order g^4 for the potential in the Klein-NAP approximation has both non-locality in the form of momentum dependence, and a dependence on the energy. By taking the initial and final momenta on the energy shell, we remove the energy dependence, but not the non-locality. This non-locality is removed by implementing the procedure in Eq. (38). The resultant potential to order g^4 is now local [16,17], and can be used in the three- and many-body calculations. In Fig. 6 we present the phase shifts for potentials calculated to order g^4 . Included are the phase shifts for the BS, Klein-NAP, and the energy independent potentials to $O(1)$ (adiabatic) and $O(m^{-1})$ (non-adiabatic). Here again, the results of the BS and Klein-NAP are in reasonable agreement, but the energy independent approximations are substantially different. The fact that the results for the adiabatic and non-adiabatic approximations are different is an indication that the series expansion in m^{-1} is not as convergent as we would like it to be. Here again the off-shell behavior of the amplitude for the Klein-NAP and energy independent potentials are substantially different as illustrated in Fig. 7 where the Kowalski-Noyes [24,25] function is given for these potentials. As was the case with the phase shifts, the non-adiabatic approximation is substantially better than the adiabatic approximation.

In the nucleon-nucleon interaction, the πNN vertex has associated with it a form factor that is a function of the exchange pion momentum. This form factor is introduced to overcome the singular nature of the potential. For the Lagrangian under consideration the potential is not singular and there is no need for any form factors. However, to examine the effect of such a form factor on the difference between the energy dependent and energy independent potentials, we have introduced a Gaussian form factor [16] by the substitution

$$g^2 \rightarrow g^2 e^{-k^2/\Lambda^2} , \quad (41)$$

where k is the momentum of the exchanged σ meson. In Fig 8 we illustrate the changes in the phase shifts for the Klein-NAP and energy independent σ -exchange potentials as a function of the cut-off parameter Λ at a laboratory energy of 100 MeV. Here we observe that as the cut-off mass is reduced, the difference between energy dependent and energy independent solutions is also partly reduced. This suggests that the energy dependence is a short range effect, and as the form factor starts to dominate the short range behavior of the potential, the role of the energy dependence is suppressed.

V. CONCLUSION

In an attempt to understand the approximation involved in reducing the potential in the Bethe-Salpeter (BS) equation to an equivalent local potential for use in the Schrödinger equation, we have considered a series of approximations which involved: (i) The reduction of the equation from four- to three-dimensions. (ii) The elimination of negative energy states or anti-particles. (iii) The removal of energy dependence and non-locality in the potential. To maintain simplicity in the model, while maintaining some relevance to the nucleon-nucleon interaction, we have considered a scalar $\phi^2\sigma$ interaction Lagrangian and included in the potential all diagrams to order g^4 , *i.e.* single σ -exchange and crossed two σ -exchange in analogy with one- and two-pion exchanges in the nucleon-nucleon problem. To set the strength of this interaction, we required that the ϕ - ϕ system have a bound state energy comparable to the deuteron binding energy.

In reducing the two-body problem from four- to three-dimensions, we have taken advantage of the equal-time Green's function [13] and the requirement that the potential have no dependence on the relative energy [9], to derive a scheme that allows for a systematic way of improving the potential to reproduce the results of the BS equation [14]. We then found that at the one σ -exchange level, neglecting the contribution of negative energy states is not substantial, provided that the ϕ - ϕ system has a weakly bound state as in the deuteron. The elimination of negative energy states gives us the time-ordered perturbation theory that has been used as the starting point for the recent derivation of the nucleon-nucleon potential [2,16,17]. At this level there is good agreement between the BS equation and the corresponding three-dimensional equation we refer to as the Klein-NAP [9]. However, the potential now is both energy dependent, *i.e.* has retardation, and is non-local. These features give rise to problems if the potential is to be used for three- and many-body calculations. To avoid such problems, it has become standard practice to remove the energy dependence and non-locality. Here we find that the approximations required to remove this energy dependence and non-locality results in a drastic change in the phase shifts.

If we adopt the view that the coupling constant can, at the local potential level, be adjusted to fit the experimental phase shifts, then the coupling constant defined at the covariant level could change by as much as 10-15%. This raises a problems in the nucleon-nucleon case if the πNN coupling constant extracted from the experimental data is to be compared with theoretical results extracted from QCD. This problem becomes central when the fit to the experimental phase shifts is at a level where the χ^2 per data is near one. More interesting is the fact that the off-shell behavior of the final local potential is substantially different from the original potential derived from the time-ordered perturbation theory. This change in off-shell behavior has significant ramifications in the three- and many-body results based on local potentials, and can be the origin of the three-nucleon force introduced to get the correct binding energy of light nuclei.

Although we have considered one of an infinite set of 3-D equations with the kernel calculated to order g^4 , we should point out that for the bound state problem with the value of the coupling constant used here, the results of the Klein equation are in good agreement with the solution of the BS equation in which the kernel is calculated to all orders [14,27,28]. Finally, we should point out that there have been similar analyses based on the Blankenbecler-Sugar equation [29,30].

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REFERENCES

- [1] C. Ordóñez, L. Ray and U. van Kolck, Phys. Rev. C **53**, 2086 (1996).
- [2] R. Machleidt, K. Holinde, and Ch. Elster, Phys. Rep. **149**, 1 (1987); R. Machleidt, in *Advances in Nuclear Physics*, edited by J. W. Negele and E. Vogt (Plenum, New York, 1989), Vol. 19, pp. 189-376.
- [3] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart, Phys. Rev. C **49**, 2950 (1994).
- [4] E. Salpeter and H. Bethe, Phys. Rev. **84**, 1232 (1951).
- [5] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
- [6] A. Klein, and T.-S. H. Lee, Phys. Rev. D **10**, 4308 (1974).
- [7] R. Blankenbecler, and R. Sugar, Phys. Rev. **142**, 1051 (1966).
- [8] F. Gross, Phys. Rev. C **26**, 2203 (1982).
- [9] A. Klein, Phys. Rev. **90**, 1101 (1953).
- [10] B.D. Keister and W.N. Polyzou, in *Advances in Nuclear Physics*, Eds. J.W. Negele and E.W. Vogt (Plenum, New York), **20**, p.225 (1991).
- [11] S. Okubo, Prog. Theor. Phys. **20**, 603 (1954).
- [12] M.G. Fuda, Nucl. Phys. **A543**, 111c (1992); Ann. Phys. (N.Y.) **231**, 1 (1994); M.G. Fuda and Y. Zhang, in *Proceeding of the XIVth International Conference on Few Body Problem in Physics*, Ed. Franz Gross, AIP Conf. Proc. No. 334 (AIP, New York), 879 (1995); Phys. Rev. C **51**, 23 (1995); M.G. Fuda, Phys. Rev. C **52**, 1260 (1995); Phys. Rev. C **54**, 495 (1996).
- [13] A. A. Logunov, and A. N. Tavkhelidze, Nuovo Cimento **29** 380 (1963).
- [14] D. R. Phillips and S. J. Wallace, Phys. Rev. C **54**, 507 (1996).
- [15] C. Itzykson and J. Zuber, *Quantum Field Theory*, (McGraw-Hill, Singapore, 1985).
- [16] Th.A. Rijken, Ann. Phys. (N.Y.) **208**, 253 (1991).
- [17] Th.A. Rijken and V.G.J. Stoks, Phys. Rev. C **46**, 73 (1992); *ibid* C **46**, 102 (1992); *ibid* C **54**, 2851 (1996); *ibid* C **54**, 2869 (1996).
- [18] A. N. Kvinikhidze and B. Blankleider, Few-Body Syst. Suppl. **7**, 294 (1994).
- [19] D. R. Phillips, Ph.D. thesis, Flinders University of South Australia, 1995.
- [20] V.G.J. Stoks and Th.A. Rijken, Nucl. Phys. **A613**, 311 (1997).
- [21] A.D. Lahiff, Honours Thesis, Flinders University of South Australia (1995).
- [22] M. Sugawara and S. Okubo, Phys. Rev. **117**, 605 (1960).
- [23] G.E. Brown, A.D. Jackson, and T.T.S. Kuo, Nucl. Phys. **A133**, 481 (1969).
- [24] K. Kowalski, Phys. Rev. Lett. **15**, 798 (1965).
- [25] H.P. Noyes, Phys. Rev. Lett. **15**, 538 (1965).
- [26] R. Machleidt, F. Sammarruca, and Y. Song, Phys. Rev. C **53**, 1483 (1996).
- [27] T. Nieuwenhuis, and J.A. Tjon, Phys. Lett. **B355**, 283 (1995); Phys. Rev. Lett. **77**, 814 (1996).
- [28] T. Nieuwenhuis, Ph.D. Thesis, University of Utrecht (1995).
- [29] M. Fortes and A.D. Jackson, Nucl. Phys. **A175**, 449 (1971).
- [30] L. Müller and W. Glockle, Nucl. Phys. **B146**, 393 (1978).

FIGURES

FIG. 1. The bound state energy of the $\phi - \phi$ system in the Bethe-Salpeter, Klein, and Klein with no anti-particle as a function of the coupling strength $\frac{g^2}{4\pi}$

FIG. 2. The phase shifts for the $\phi - \phi$ system in the Bethe-Salpeter, Klein, and Klein with no anti-particles as a function of the energy for $\frac{g^2}{4\pi} = 1.646$. This corresponds to a binding energy comparable to the deuteron binding energy.

FIG. 3. The diagrams that contribute to the potential $K_1^{(4)}$ to fourth order in the coupling constant in time ordered perturbation theory. Diagrams (a) and (b) are the single σ -exchange, *i.e.* $K_1^{(2)}$, while diagrams (e) – (j) are the crossed diagrams. Here diagrams (c) and (d) are those diagrams that are in the iterated σ -exchange in the BS equations but are not generated in the iteration of the Klein equation and are part of the fourth order contribution to the potential.

FIG. 4. The s -wave phase shifts for the BS, Klein-NAP, and energy independent σ -exchange potentials.

FIG. 5. The Kowalski-Noyes half-off-shell function for the Klein-NAP and the energy independent σ -exchange potentials. The energy independent I corresponds to the potential that gave the phase shifts in Fig. 4, while energy independent II corresponds to the case when the coupling constant g is adjusted to give the same phase shifts for the energy independent and the Klein-NAP potentials.

FIG. 6. The s -wave phase shifts for the BS, Klein-NAP, adiabatic $O(1)$, non-adiabatic $O(m^{-1})$ single σ -exchange and crossed sigma potentials.

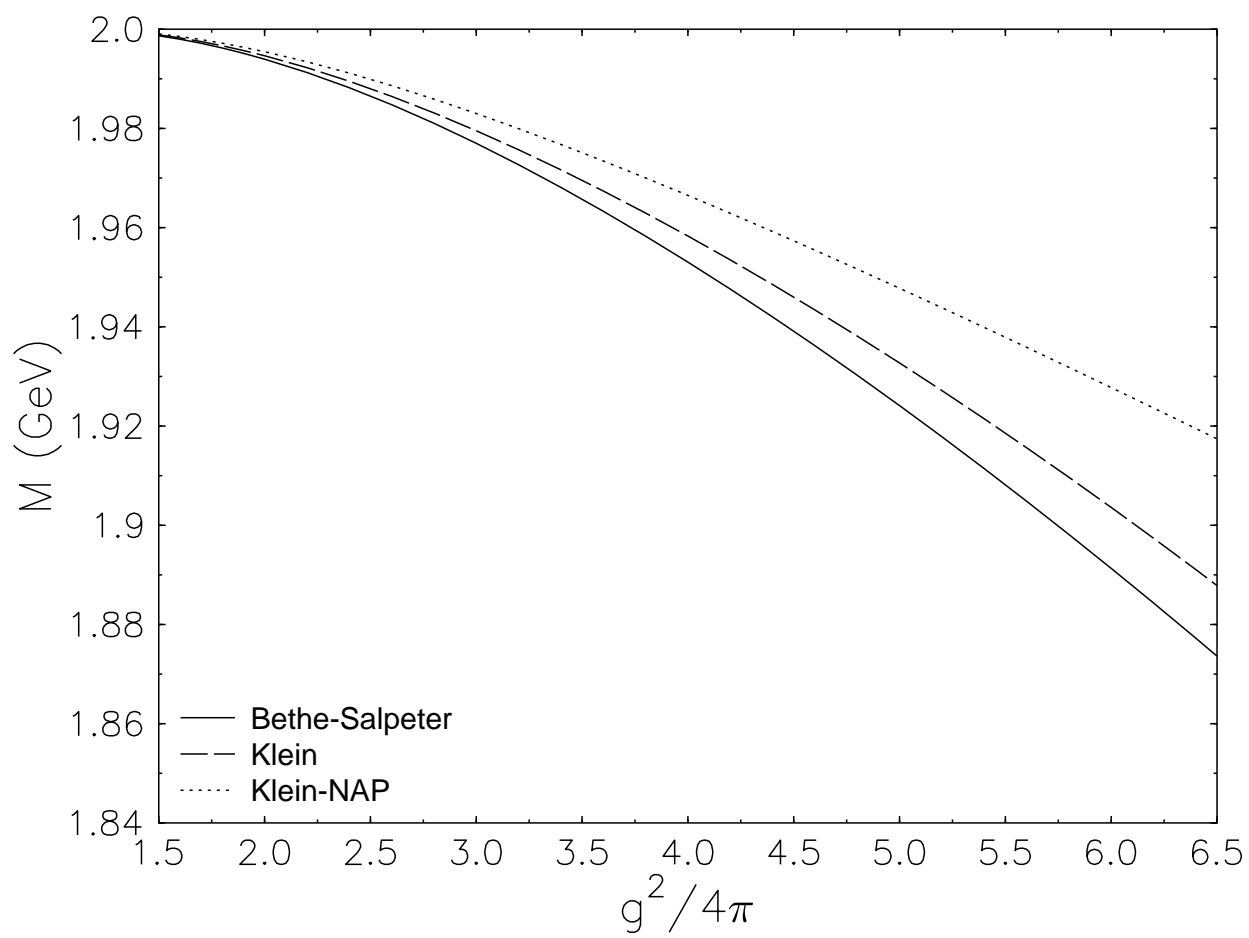
FIG. 7. The Kowalski-Noyes half-off-shell function for the Klein-NAP, adiabatic $O(1)$, non-adiabatic $O(m^{-1})$ single σ -exchange and crossed sigma potentials.

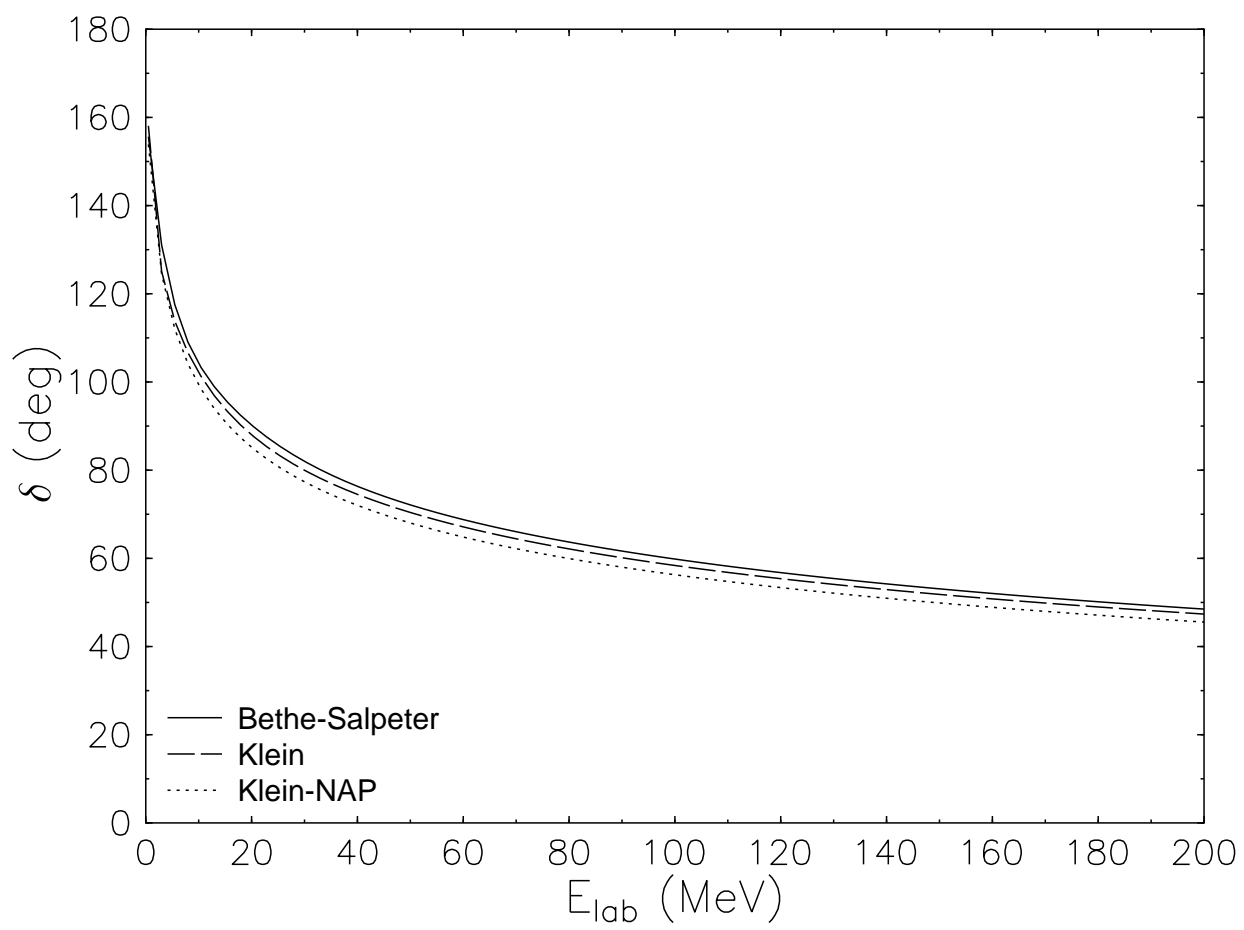
FIG. 8. The s -wave phase shifts at $E_{lab} = 100$ MeV for the Klein-NAP and energy independent potential for the σ -exchange potential.

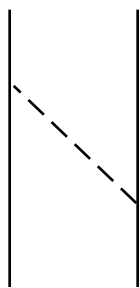
TABLES

TABLE I. Comparison of the coupling constant required to give a binding energy of 2.225 MeV for the ϕ - ϕ system.

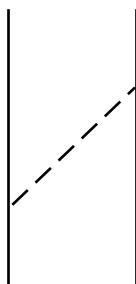
Potential	$\frac{g^2}{4\pi}$
Bethe-Salpeter	1.646
Klein-NAP	1.747
Energy Independent	1.433



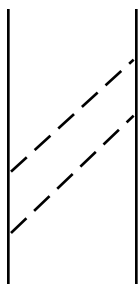




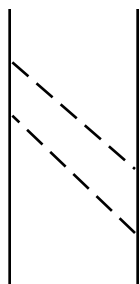
(a)



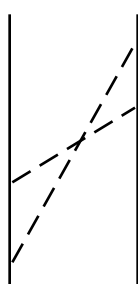
(b)



(c)



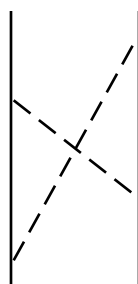
(d)



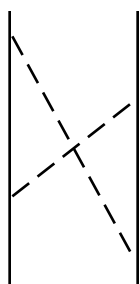
(e)



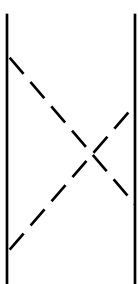
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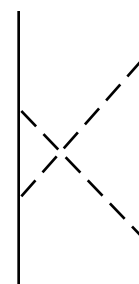
(g)



(h)



(i)



(j)

